Lecture 24

Parameterized Curves

Recall earlier in the semester when we parameterized curves using the parameter, t.

Ex. 1 Find a parameterization of a circle of radius, r.

$$x = r\cos(t) \qquad y = r\sin(t)$$

$$\bar{r} = \langle r\cos(t), r\sin(t) \rangle \qquad \text{for} \quad 0 \leq t \leq 2\pi$$

Ex. 2 Find a parameterization of a vertically oriented helix.

$$x = r\cos(t) \qquad y = r\sin(t) \qquad z = t \qquad \forall t$$

$$\bar{r} = \langle r\cos(t), r\sin(t), t \rangle$$

In general we can parameterize curves by

$$r(t) = \langle x(t), y(t), z(t) \rangle$$

Parameterized Surfaces

With one parameter, we could "sweep out" 1-D curves. If we use a second, we can sweep out surfaces.

If a surface Σ can be written as,

$$\bar{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle,$$

We call Σ a parameterized surface. The function $\bar{r}(u,v)$ takes a regin R and maps it to Σ.

Ex. 3 Find a parameterization of a sphere with radius a.

$$x = \rho\sin(\varphi)\cos(\theta) \qquad y = \rho\sin(\varphi)\sin(\theta) \qquad z = \rho\cos(\varphi)$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = a$$

$$\bar{r}(\varphi, \theta) = \langle a\sin(\varphi)\cos(\theta), a\sin(\varphi)\sin(\theta), a\cos(\varphi) \rangle$$

$$\text{for} \quad 0 \leq \varphi \leq \pi \quad \& \quad 0 \leq \theta \leq 2\pi$$

Ex. 4 Parameterize the surface that is the part of the cylinder $x^2+y^2=9$ between $z=\emptyset$ & $z=2$.

The cylinder can be parameterized as

$$\bar{r}(\theta) = \langle 3\cos(\theta), 3\sin(\theta) \rangle$$

To include the z component we add

$$\bar{r}(\theta, z) = \langle 3\cos(\theta), 3\sin(\theta), z \rangle$$

Surface Area

Let Σ be a smooth parameterized surface given by

$$\bar{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

on vertically or horizontally simple region, R. The the surface area S of Σ is given by,

$$S = \iint_R \| \bar{r}_u(u,v) \times \bar{r}_v(u,v) \| \, dA$$

where $\bar{r}_u(u,v) = \langle \frac{\partial}{\partial u} x(u,v), \frac{\partial}{\partial u} y(u,v), \frac{\partial}{\partial u} z(u,v) \rangle$

Ex. 5 Find the surface of a sphere of radius a.

$$\bar{r}(\varphi, \theta) = \langle a\sin(\varphi)\cos(\theta), a\sin(\varphi)\sin(\theta), a\cos(\varphi) \rangle$$

$$\bar{r}_\varphi(\varphi, \theta) = \langle a\cos(\varphi)\cos(\theta), a\cos(\varphi)\sin(\theta), -a\sin(\varphi) \rangle$$

$$\bar{r}_\theta(\varphi, \theta) = \langle -a\sin(\varphi)\sin(\theta), a\sin(\varphi)\cos(\theta), \emptyset \rangle$$

$$\bar{r}_\varphi \times \bar{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos(\varphi)\cos(\theta) & a\cos(\varphi)\sin(\theta) & -a\sin(\varphi) \\ -a\sin(\varphi)\sin(\theta) & a\sin(\varphi)\cos(\theta) & \emptyset \end{vmatrix}$$

$$\bar{r}_\varphi \times \bar{r}_\theta = \langle a^2\sin^2(\varphi)\cos(\theta), a^2\sin^2(\varphi)\sin(\theta), a^2\cos(\varphi)\sin(\varphi) \rangle$$

$$|\bar{r}_\varphi \times \bar{r}_\theta| = \sqrt{a^4\sin^4(\varphi)\left[\cos^2(\theta) + \sin^2(\theta)\right] + a^4\cos^2(\varphi)\sin^2(\varphi)} = a^2\sin(\varphi)$$

$$S = \iint_R a^2 \sin(\varphi)\, dA$$

$$= \int_{\emptyset}^{\pi} \int_{0}^{2\pi} a^2 \sin(\varphi)\, d\theta\, d\varphi$$

$$= 2\pi a^2 \int_{\emptyset}^{\pi} \sin(\varphi)\, d\varphi$$

$$= -2a^2\pi \cos(\varphi)\Big|_{\emptyset}^{\pi}$$

$$S = 4\pi a^2$$